

# Multispecies quantum Hurwitz numbers<sup>\*</sup>

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## Abstract

The construction of hypergeometric 2D Toda  $\tau$ -functions as generating functions for quantum Hurwitz numbers is extended here to multispecies families. Both the enumerative geometrical significance of these multispecies weighted Hurwitz numbers as weighted enumerations of branched coverings of the Riemann sphere and their combinatorial significance in terms of weighted paths in the Cayley graph of  $S_n$  are derived.

## 1 Introduction

### 1.1 Weighted Hurwitz numbers

Quantum weighted Hurwitz numbers were introduced in [5], in four variants, and a 1-parameter family of 2D Toda  $\tau$ -function generating functions was constructed for each. In the first case, the weight generating function is:

$$E(q, z) := \prod_{i=0}^{\infty} (1 + q^i z) = 1 + \sum_{i=1}^{\infty} E_i(q) z^i, \quad (1.1)$$

$$E_i(q) := \left( \prod_{j=1}^j \frac{q^{j-1}}{1 - q^j} \right). \quad (1.2)$$

The second is a slight modification of this, with weight generating function

$$E'(q, z) := \prod_{i=1}^{\infty} (1 + q^i z) = 1 + \sum_{i=1}^{\infty} E'_i(q) z^i, \quad (1.3)$$

$$E'_i(q) := \left( \prod_{j=1}^j \frac{q^i}{1 - q^j} \right). \quad (1.4)$$

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The third case is based on the weight generating function

$$H(q, z) := \prod_{i=1}^{\infty} (1 - q^i z)^{-1} = 1 + \sum_{i=1}^{\infty} H_i(q) z^i, \quad (1.5)$$

$$H_i(q) := \left( \prod_{j=1}^i \frac{1}{1 - q^j} \right), \quad (1.6)$$

and the final case is a hybrid, formed from the product of the first and third for two distinct quantum deformation parameters  $q$  and  $p$ , with weight generating function

$$Q(q, p, z) := \prod_{k=0}^{\infty} (1 + q^k z)(1 - p^k z)^{-1} = \sum_{i=0}^{\infty} Q_i(q, p) z^i, \quad (1.7)$$

$$Q_i(q, p) := \sum_{m=0}^i q^{\frac{1}{2}m(m-1)} \left( \prod_{j=1}^m (1 - q^j) \prod_{j=1}^{i-m} (1 - p^j) \right)^{-1}, \quad Q_{\lambda}(q, p) = \prod_{i=1}^{\ell(\lambda)} Q_{\lambda_i}(q, p), \quad (1.8)$$

These are all expressible as exponentials of the quantum dilogarithm function

$$\text{Li}_2(q, z) := \sum_{k=1}^{\infty} \frac{z^k}{k(1 - q^k)}. \quad (1.9)$$

$$E(q, z) = e^{-\text{Li}_2(q, -z)}, \quad E'(q, z) = (1 + z)^{-1} e^{-\text{Li}_2(q, -z)} \quad (1.10)$$

$$H(q, z) = e^{\text{Li}_2(q, z)}, \quad Q(q, p, z) = e^{\text{Li}_2(p, z) - \text{Li}_2(q, -z)}, \quad (1.11)$$

The content products for the first and third of these are defined to be

$$r_{\lambda}^{E(q, z)}(N) := \prod_{(ij) \in \lambda} E(q, (N + j - i)z) \quad (1.12)$$

$$r_{\lambda}^{H(q, z)}(N) := \prod_{(ij) \in \lambda} H(q, (N + j - i)z). \quad (1.13)$$

The associated hypergeometric 2D Toda  $\tau$ -functions have diagonal double Schur function expansions with these as coefficients:

$$\tau^{E(q, z)}(N, \mathbf{t}, \mathbf{s}) = \sum_{\lambda} r_{\lambda}^{E(q, z)}(N) S_{\lambda}(\mathbf{t}) S_{\lambda}(\mathbf{s}) \quad (1.14)$$

$$\tau^{H(q, z)}(N, \mathbf{t}, \mathbf{s}) = \sum_{\lambda} r_{\lambda}^{H(q, z)}(N) S_{\lambda}(\mathbf{t}) S_{\lambda}(\mathbf{s}). \quad (1.15)$$

Using the Frobenius character formula [11],

$$S_{\lambda} = \sum_{\mu, |\mu|=|\lambda|} \frac{\chi_{\lambda} P_{\mu}}{Z_{\mu}} \quad (1.16)$$

and setting  $N = 0$ , these may be rewritten as double expansions in the power sum symmetric functions [5]:

$$\tau^{E(q,z)}(\mathbf{t}, \mathbf{s}) := \tau^{E(q,z)}(0, \mathbf{t}, \mathbf{s}) = \sum_{d=0}^{\infty} z^d \sum_{\mu, \nu, |\mu|=|\nu|} H_{E(q)}^d(\mu, \nu) P_{\mu}(\mathbf{t}) P_{\nu}(\mathbf{s}), \quad (1.17)$$

$$\tau^{H(q,z)}(\mathbf{t}, \mathbf{s}) := \tau^{H(q,z)}(0, \mathbf{t}, \mathbf{s}) = \sum_{d=0}^{\infty} z^d \sum_{\mu, \nu, |\mu|=|\nu|} H_{H(q)}^d(\mu, \nu) P_{\mu}(\mathbf{t}) P_{\nu}(\mathbf{s}). \quad (1.18)$$

The coefficients are the corresponding quantum Hurwitz numbers  $H_{E(q)}^d(\mu, \nu)$ ,  $H_{H(q)}^d(\mu, \nu)$ , which count weighted  $n$ -sheeted branched coverings of the Riemann sphere, defined by

$$H_{E(q)}^d(\mu, \nu) := \sum_{k=0}^{\infty} \sum'_{\substack{\mu^{(1)}, \dots, \mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)}) = d}} W_{E(q)}(\mu^{(1)}, \dots, \mu^{(k)}) H(\mu^{(1)}, \dots, \mu^{(k)}, \mu, \nu), \quad (1.19)$$

$$H_{H(q)}^d(\mu, \nu) := \sum_{k=0}^{\infty} (-1)^{k+d} \sum'_{\substack{\mu^{(1)}, \dots, \mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)}) = d}} W_{H(q)}(\mu^{(1)}, \dots, \mu^{(k)}) H(\mu^{(1)}, \dots, \mu^{(k)}, \mu, \nu). \quad (1.20)$$

where the weightings for such covers with  $k$  additional branch points are:

$$W_{E(q)}(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} \frac{q^{(k-1)\ell^*(\mu^{(1)})} \dots q^{\ell^*(\mu^{(k-1)})}}{(1 - q^{\ell^*(\mu^{(\sigma(1))})}) \dots (1 - q^{\ell^*(\mu^{(\sigma(k))})})}, \quad (1.21)$$

$$W_{H(q)}(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} \frac{1}{(1 - q^{\ell^*(\mu^{(\sigma(1))})}) \dots (1 - q^{\ell^*(\mu^{(\sigma(k))})})}. \quad (1.22)$$

Here

$$\ell^*(\mu) := |\mu| - \ell(\mu) \quad (1.23)$$

is the *colength* of the partition  $\mu$ , which is the index of coalescence of the sheets of the branched cover over the branch points with ramification profiles  $\mu^{(i)}$ , the sum  $\sum'_{\substack{\mu^{(1)}, \dots, \mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)}) = d}}$

is over all  $k$ -tuples of partitions having nontrivial ramification profiles that satisfy the constraint  $\sum_{i=1}^k \ell^*(\mu^{(i)}) = d$ , and  $H(\mu^{(1)}, \dots, \mu^{(k)}, \mu, \nu)$  is the number of branched  $n$ -sheeted coverings, up to isomorphism, having  $k+2$  branch points with ramification profiles  $(\mu^{(1)}, \dots, \mu^{(k)}, \mu, \nu)$ .

These thus count weighted covers with a pair of branch points, say  $(0, \infty)$ , having ramification profiles of type  $(\mu, \nu)$  and an arbitrary number of further branch points, whose profiles  $(\mu^{(1)}, \dots, \mu^{(k)})$  are constrained only by the requirement that the sum of the colengths, which is related to the genus by the Riemann-Hurwitz formula

$$\sum_{i=1}^k \ell^*(\mu^{(i)}) = 2g - 2 + \ell(\mu) + \ell(\nu) = d, \quad (1.24)$$

be fixed to equal  $d$ .

Another interpretation that is purely combinatorial can be given to the quantum Hurwitz numbers  $H_{E(q)}^d(\mu, \nu)$  and  $H_{H(q)}^d(\mu, \nu)$  appearing in (1.18), as follows. Let  $(a_1b_1) \cdots (a_db_d)$  be a product of  $d$  transpositions  $(a_ib_i) \in S_n$  in the symmetric group  $S_n$  with  $a_i < b_i$ ,  $i = 1, \dots, d$ . If  $h \in S_n$  is in the conjugacy class  $\text{cyc}(\mu) \subset S_n$  consisting of elements with cycle lengths equal to the parts  $(\mu_1, \dots, \mu_{\ell(\mu)})$  of the partition  $\mu$  of weight  $|\mu| = n$ , and length  $\ell(\mu)$ , we may view the successive steps in the product

$$(a_1b_1) \cdots (a_db_d)h \quad (1.25)$$

as a path in the Cayley graph generated by all transpositions. To each such path, we attach a *signature* consisting of a partition  $\lambda$  of  $d$ ,  $|\lambda| = d$ , whose parts  $\lambda_i$  consist of the number of transpositions  $(a_ib_i)$  sharing the same second element. If we further require that the ones with equal second elements be grouped together into consecutive subsequences in which these second elements are constant, with the consecutive subsequences strictly increasing in their second elements, then the number  $\tilde{N}_\lambda$  of elements with signature  $\lambda$  is related to the number  $N_\lambda$  of such ordered sequences by

$$\tilde{N}_\lambda = \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} N_\lambda \quad (1.26)$$

Denote the number of such paths from the conjugacy class of cycle type  $\text{cyc}(\mu)$  to the one of type  $\text{cyc}(\nu)$  having signature  $\lambda$  as  $\tilde{m}_{\mu\nu}^\lambda$ , and the number of ordered sequences of type  $\lambda$  as  $m_{\mu\nu}^\lambda$ . Thus

$$\tilde{m}_{\mu\nu}^\lambda = \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} m_{\mu\nu}^\lambda. \quad (1.27)$$

For all paths of signature  $\lambda$  we assign weights

$$\tilde{E}_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \lambda_i! E_{\lambda_i}(q) = \prod_{i=1}^{\ell(\lambda)} \frac{\lambda_i! q^{\frac{1}{2}\lambda_i(\lambda_i-1)}}{\prod_{j=1}^{\lambda_i} (1-q^j)}, \quad (1.28)$$

$$\tilde{H}_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \lambda_i! H_{\lambda_i}(q) = \prod_{i=1}^{\ell(\lambda)} \frac{\lambda_i!}{\prod_{j=1}^{\lambda_i} (1-q^j)} \quad (1.29)$$

to paths in the Cayley graph, and a pair of corresponding combinatorial weighted Hurwitz numbers

$$F_{E(q)}^d(\mu, \nu) := \frac{1}{d!} \sum_{\lambda, |\lambda|=d} \tilde{E}_\lambda \tilde{m}_{\mu\nu}^\lambda, \quad (1.30)$$

$$F_{H(q)}^d(\mu, \nu) := \frac{1}{d!} \sum_{\lambda, |\lambda|=d} \tilde{H}_\lambda \tilde{m}_{\mu\nu}^\lambda, \quad (1.31)$$

that give the weighted enumeration of paths, using the weighting factors  $\tilde{E}_\lambda(q)$  and  $\tilde{H}_\lambda(q)$  respectively for all paths of signature  $\lambda$ .

It was shown in [5] that the enumerative geometrical and combinatorial definitions of these quantum weighted Hurwitz numbers coincide:

$$H_{E(q)}^d(\mu, \nu) = F_{E(q)}^d(\mu, \nu), \quad H_{H(q)}^d(\mu, \nu) = F_{H(q)}^d(\mu, \nu). \quad (1.32)$$

A similar result holds for weights generated by the function  $E'(q, z)$ , with corresponding quantum Hurwitz numbers defined by

$$H_{E'(q)}^d(\mu, \nu) := \sum_{k=0}^{\infty} \sum'_{\substack{\mu^{(1)}, \dots, \mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)}) = d}} W_{E'(q)}(\mu^{(1)}, \dots, \mu^{(k)}) H(\mu^{(1)}, \dots, \mu^{(k)}, \mu, \nu), \quad (1.33)$$

where the weights  $W_{E'(q)}(\mu^{(1)}, \dots, \mu^{(k)})$  are given by

$$W_{E'(q)}(\mu^{(1)}, \dots, \mu^{(k)}) := \frac{1}{k!} \sum_{\sigma \in S_k} \frac{q^{(k)\ell^*(\mu^{(1)})} \dots q^{\ell^*(\mu^{(k)})}}{(1 - q^{\ell^*(\mu^{(\sigma(1))})}) \dots (1 - q^{\ell^*(\mu^{(\sigma(1))})} \dots q^{\ell^*(\mu^{(\sigma(k))})})}, \quad (1.34)$$

$$:= \frac{1}{k!} \sum_{\sigma \in S_k} \frac{1}{(q^{-\ell^*(\mu^{(\sigma(1))})} - 1) \dots (q^{-\ell^*(\mu^{(\sigma(1))})} \dots q^{-\ell^*(\mu^{(\sigma(k))})} - 1)}. \quad (1.35)$$

Choosing  $q$  as a positive real number, parametrizing it as

$$q = e^{-\beta \hbar \omega}, \quad \beta = \frac{1}{kT} \quad (1.36)$$

and identifying the energy levels  $\epsilon_k$  as those for a Bose gas with linear spectrum in the integers, as for a 1-D harmonic oscillator

$$\epsilon_k := k \hbar \omega, \quad k \in \mathbf{N}, \quad (1.37)$$

we see that if we assign the energy

$$\epsilon(\mu) := \epsilon_{\ell^*(\mu)} = \ell^*(\mu) \hbar \omega \quad (1.38)$$

to each branch point with ramification profile of type  $\mu$ , it contributes a factor

$$n(\mu) = \frac{1}{e^{\beta \epsilon(\mu)} - 1} \quad (1.39)$$

to the weighting distributions, the same as that for a bosonic gas.

## 2 2D Toda $\tau$ -functions as generating functions for multispecies weighted Hurwitz numbers

### 2.1 The multiparameter family of $\tau$ -functions $\tau^{G(\mathbf{z}, \mathbf{w})}(N, \mathbf{t}, \mathbf{s})$

We now consider a multiparameter family of weight generating functions  $G(\mathbf{z}; \mathbf{w})$  obtained by taking the product of any number of generating functions  $G_i(z_i)$  and  $\tilde{G}_j(w_j)$  for distinct sets of generating function parameters  $\mathbf{z} = (z_1, \dots, z_l)$ ,  $\mathbf{w} = (w_1, \dots, w_m)$ .

$$Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w}) := \prod_{i=1}^l E(q_i, z_i) \prod_{j=1}^m H(p_j, w_j). \quad (2.1)$$

Following the approach developed in [5], we define an associated element of the center  $\mathbf{Z}(\mathbf{C}[S_n])$  of the group algebra  $\mathbf{C}[S_n]$  by

$$\hat{Q}(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w}, \mathcal{J}) := \prod_{a=1}^n Q(\mathbf{q}, \mathcal{J}_a \mathbf{z}; \mathbf{p}, \mathcal{J}_a \mathbf{w}), \quad (2.2)$$

where  $\mathcal{J} := (\mathcal{J}_1, \dots, \mathcal{J}_n)$  are the Jucys-Murphy elements [9, 12, 1]

$$\mathcal{J}_1 := 0, \quad \mathcal{J}_b := \sum_{a=1}^{b-1} (ab), \quad b = 1, \dots, n, \quad (2.3)$$

which generate an abelian subalgebra of  $\mathbf{Z}(\mathbf{C}[S_n])$ . The element  $\hat{Q}(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w}, \mathcal{J})$  defines an endomorphism of  $\mathbf{Z}(\mathbf{C}[S_n])$  under multiplication, which is diagonal in the basis  $\{F_\lambda\}$  of  $\mathbf{Z}(\mathbf{C}[S_n])$  consisting of the orthogonal idempotents, corresponding to irreducible representations, labelled by partitions  $\lambda$  of  $n$ :

$$\hat{Q}(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w}, \mathcal{J}) F_\lambda = r_\lambda^{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})} F_\lambda \quad (2.4)$$

where

$$r_\lambda^{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})} = \prod_{i=1}^l r_\lambda^{E(q_i)}(z_i) \prod_{j=1}^m r_\lambda^{H(p_j)}(w_j). \quad (2.5)$$

More generally, defining

$$r_\lambda^{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})}(N) = \prod_{i=1}^l r_\lambda^{E(q_i)}(N, z_i) \prod_{j=1}^m r_\lambda^{H(p_j)}(N, w_j), \quad (2.6)$$

where

$$r_\lambda^{E(q)}(N, z) := \prod_{(ij) \in \lambda} E(q, (N + j - i)z) \quad (2.7)$$

$$r_\lambda^{H(q)}(N, z) := \prod_{(ij) \in \lambda} H(q, (N + j - i)z), \quad (2.8)$$

we have

$$r_\lambda^{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})} = r_\lambda^{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})}(0). \quad (2.9)$$

From general considerations [14, 8], the double Schur function series

$$\tau^{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})}(N, \mathbf{t}, \mathbf{s}) := \sum_{\lambda} r_\lambda^{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})}(N) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}) \quad (2.10)$$

is known to define a family of  $2D$  Toda  $\tau$ -functions of hypergeometric type.

## 2.2 Multispecies geometric weighted Hurwitz numbers

We now consider coverings in which the branch points are divided into two different classes,  $\{\mu_+^{(i, u_i)}\}_{i=1, \dots, l; u_i=1, \dots, k_i^-}$  and  $\{\mu_-^{(j, v_j)}\}_{j=1, \dots, m; v_j=1, \dots, k_j^-}$  corresponding to the weight generating functions of type  $E(q_i)$  and  $H(p_j)$  respectively, each of which is further divided into  $l$  and  $m$  distinct species (or “colours”), of which there are  $\{k_i^+\}$  and  $\{k_j^-\}$  branch points of types  $E$  and  $H$  and colours  $i$  and  $j$  respectively. The weighted number of such coverings with specified total colengths  $\mathbf{c} = (c_1, \dots, c_l)$ ,  $\mathbf{d} = (d_1, \dots, d_m)$ ,  $c_i, d_j \in \mathbf{N}$  for each class and colour is the multispecies quantum Hurwitz number

$$\begin{aligned} H_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})}(\mu, \nu) := & \sum_{\substack{\{k_i^+=1; k_j^-=1\} \\ i=1, \dots, l; j=1, \dots, m}}^{\infty} \sum_{\substack{\{\mu_+^{(i, u_i)}\}; \mu_-^{(j, v_j)}\} \\ \sum_{u_i=1}^{k_i^+} \ell^*(\mu_+^{(i, u_i)}) = c_i, \sum_{v_j=1}^{k_j^-} \ell^*(\mu_-^{(j, v_j)}) = d_j}} \\ & \times \prod_{i=1}^l W_{E(q_i)}(\mu_+^{(i, 1)}, \dots, \mu_+^{(i, k_i^+)}) \prod_{j=1}^m W_{H(p_j)}(\mu_-^{(j, 1)}, \dots, \mu_-^{(j, k_j^-)}) H(\{\mu_+^{(i, u_i)}; \mu_-^{(j, v_j)}\}, \mu, \nu). \end{aligned} \quad (2.11)$$

Substituting the Frobenius-Schur formula (??) and the Frobenius character formula (1.16) into (2.10), it follows that  $\tau_{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})}(N, \mathbf{t}, \mathbf{s})$  is the generating function for  $H_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})}(\mu, \nu)$ . Using multi-index notation to denote

$$\prod_{i=1}^l z_i^{c_i} \prod_{j=1}^m w_j^{d_j} =: \mathbf{z}^{\mathbf{c}} \mathbf{w}^{\mathbf{d}}, \quad (2.12)$$

we have:

**Theorem 2.1.**

$$\tau^{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})}(0, \mathbf{t}, \mathbf{s}) := \sum_{\substack{(\infty, \dots, \infty) \\ \mathbf{c}=(0, \dots, 0); \mathbf{d}=(0, \dots, 0)}} \mathbf{z}^{\mathbf{c}} \mathbf{w}^{\mathbf{d}} \sum_{\mu, \nu} H_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})}(\mu, \nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}). \quad (2.13)$$

## 2.3 Multispecies combinatorial weighted Hurwitz numbers

Another basis for  $\mathbf{Z}(\mathbf{C}[S_n])$  consists of the cycle sums

$$C_\mu := \sum_{h \in \text{cyc}(\mu)} h, \quad (2.14)$$

where  $\text{cyc}(\mu)$  denotes the conjugacy class of elements  $h \in \text{cyc}(\mu)$  with cycle lengths equal to the parts of  $\mu$ . The two are related by

$$F_\lambda = h_\lambda^{-1} \sum_{\mu, |\mu|=|\lambda|} \chi_\lambda(\mu) C_\mu, \quad (2.15)$$

where  $\chi_\lambda(\mu)$  denotes the irreducible character of the irreducible representation of type  $\lambda$  evaluated on the conjugacy class  $\text{cyc}(\mu)$ . Under the characteristic map, this is equivalent to the Frobenius character formula (1.16). The Macmahon generating function

$$\prod_{i=1}^{\infty} (1 - q^i)^{-1} = \sum_{n=0}^{\infty} D_n q^n, \quad (2.16)$$

gives the number  $D_n$  of partiitions of  $n$ . We denote by  $\mathbf{F}_{E(q_i)}^{(n, c_i)}$  and  $\mathbf{F}_{H(q_j)}^{(n, d_j)}$  the  $D_n \times D_n$  matrices, whose elements are  $\left(F_{E(q_i)}^{c_i}(\mu, \nu)\right)_{|\mu|=|\nu|=n}$  and  $\left(F_{H(q_j)}^{d_j}(\mu, \nu)\right)_{|\mu|=|\nu|=n}$ , respectively, for  $i = 1, \dots, l$ ,  $j = 1, \dots, m$ . Since these represent central elements of the group algebra  $\mathbf{C}[S_n]$ , they commute amongst themselves. Denoting the matrix product:

$$\mathbf{F}_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})} = \prod_{i=1}^l \mathbf{F}_{E(q_i)}^{(n, c_i)} \prod_{j=1}^m \mathbf{F}_{H(q_j)}^{(n, d_j)}, \quad (2.17)$$

its matrix elements, denoted  $F_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})}(\mu, \nu)$ , may be interpreted as the weighted number of

$$d := \sum_{i=1}^l c_i + \sum_{j=1}^m d_j \quad (2.18)$$

step paths in the Cayley graph from the conjugacy class of cycle type  $\mu$  to the one of type  $\nu$ , where all paths are divided into equivalence classes, according to their multisignature. This consists of partitions of  $\lambda_+^{(i)}$  of  $c_i$  and  $\lambda_-^{(j)}$  of  $d_j$  with

$$\lambda_+^{(i)} = \ell(\lambda_+^{(i)}), \quad \lambda_-^{(j)} = \ell(\lambda_-^{(j)}) \quad (2.19)$$

parts  $\lambda_+^{(i)}$  and  $\lambda_-^{(j)}$ , each of which is itself subdivided into parts  $(\lambda_{+, u_i}^{(i)}, \lambda_{-, v_j}^{(j)})$ , in which the second elements of the transpositions are constant, and distinct for each  $\lambda_{+, u_i}^{(i)}$ , or  $\lambda_{-, v_j}^{(j)}$ . We



define the hypersignature of such a path as the set of numbers  $\{\lambda_{+,u_i}^{(i)}, \lambda_{-,v_j}^{(j)}\}_{u_i=1,\dots,k_i^+, v_j=1,\dots,k_j^-}$ . The weight for each path within such an equivalence class is then the product of the weights for each subsegment:

$$\prod_{i=1}^l \prod_{u_i=1}^{k_i^+} E(q_i)_{\lambda_{(+,u_i)}^{(i)}} \prod_{j=1}^m \prod_{v_j=1}^{k_j^-} H(p_j)_{\lambda_{(-,v_j)}^{(j)}} \quad (2.20)$$

and  $F_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})}(\mu, \nu)$  is the sum of these, each multiplied by the number of elements of the equivalence class of paths. with the given hypersignature. The multispecies generalization of (1.32) is equality of the geometric and combinatorial Hurwitz numbers:

**Theorem 2.2.**

$$F_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})}(\mu, \nu) = H_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})}(\mu, \nu). \quad (2.21)$$

*Proof.* Applying the central element (2.4) to the cycle sum  $C_\mu$  gives

$$\hat{Q}(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w}, \mathcal{J}) C_\mu = \sum_{\nu, |\nu|=|\mu|} F_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})}(\mu, \nu) C_\nu \quad (2.22)$$

and the orthogonality of group characters that

$$\tau^{Q(\mathbf{q}, \mathbf{z}; \mathbf{p}, \mathbf{w})}(\mathbf{t}, \mathbf{s}) := \sum_{\mathbf{c}=(0,\dots,0); \mathbf{d}=(0,\dots,0)}^{(\infty,\dots,\infty)} \mathbf{z}^{\mathbf{c}} \mathbf{w}^{\mathbf{d}} \sum_{\mu, \nu} F_{(\mathbf{q}, \mathbf{p})}^{(\mathbf{c}, \mathbf{d})}(\mu, \nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}). \quad (2.23)$$

Comparing this with eq. (2.13) proves the result. □

### Remark 2.1. Multispecies Bose gases

Returning to the interpretation of the quantum Hurwitz weighting distributions in terms of Bose gases, if we choose each  $q_i$  to be a positive real number with  $q_i < 1$ , and parametrize it, as before,

$$q_i = e^{-\beta \hbar \omega_i} \quad (2.24)$$

for some ground state energy  $\hbar \omega_i$ , and again choose a linear energy spectrum, with energy assigned to the branchpoint  $\mu$  of type  $i$  with profile type  $\mu$  to be

$$\epsilon^{(i)}(\mu) := \ell^*(\mu) \hbar \omega_i \quad (2.25)$$

we see that the resulting contributions to the weighting distributions from each species of branch points of ramification type  $\mu^{(j)}$  are given by

$$n_B^{(i)}(\mu) = \frac{1}{e^{\beta \epsilon^{(i)}(\mu)} - 1}, \quad (2.26)$$

which are those of a multispecies mixture of Bose gases.

## 2.4 Examples of general multispecies Hurwitz numbers

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